duces that $A_n \subseteq U_G$. Let (c_k, d_k) denote the components of U_G. By the lemma one has $G(c_k) \geq G(d_k)$ for each k and hence $m^*(E \cap (c_k, d_k)) \leq n(d_k - c_k)/(n+1)$. By (P2), (P1), and (P3) one thus obtains

$$m^*(A_n) \leq \sum_k m^*(A_n \cap (c_k, d_k)) \leq \sum_k \frac{n}{n+1}(d_k - c_k) = \frac{n}{n+1} m^*(U_G).$$

Therefore $m^*(A_n) < n(m^*(A_n) + \varepsilon)/(n+1)$, which implies that $m^*(A_n) < n\varepsilon$. The assertion follows because ε is arbitrary. ∎

By symmetry, the set $B := \{x \in E / \underline{d}_-(E, x) < 1\}$ also has outer measure zero. Hence $\underline{d}_+(E, x) = \underline{d}_-(E, x) = 1$ for almost all $x \in E$, and the proof of Lebesgue's theorem is complete.

REFERENCES

1. H. Lebesgue, *Leçons sur l'Intégration et la Recherche des Fonctions Primitives*, Gauthier Villars, Paris, 1904.
2. B. Maurey and J.-P. Tacchi, A propos du théorème de densité de Lebesgue, *Travaux mathématiques, Fasc. IX*, Sém. Math. Luxembourg, Luxembourg, 1997, pp. 1–21.
3. F. Riesz, Sur l'existence de la dérivée des fonctions monotones et sur quelques problèmes qui s'y rattachent, *Acta Sci. Math.* **5** (1930–1932) 208–221.
4. A. C. M. van Rooij and W. H. Schikhof, *A Second Course on Real Functions*, Cambridge University Press, Cambridge, 1982.
5. W. Sierpiński, Démonstration élémentaire du théorème sur la densité des ensembles, *Fund. Math.* **4** (1923) 167–171.
6. L. Zajíček, An elementary proof of the one-dimensional density theorem, *Amer. Math. Monthly* **86** (1979) 297–298.

Lycée cantonal de Porrentruy, place Blarer-de-Wartensee 2, CH-2900 Porrentruy, Switzerland
cafaure@bluewin.ch

A Simple Proof of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ and Related Identities

Josef Hofbauer

1. A PROOF FOR

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \tag{1}$$

Repeated application of the identity

$$\frac{1}{\sin^2 x} = \frac{1}{4\sin^2 \frac{x}{2} \cos^2 \frac{x}{2}} = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{x}{2}} + \frac{1}{\cos^2 \frac{x}{2}}\right] = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{x}{2}} + \frac{1}{\sin^2 \frac{\pi+x}{2}}\right] \tag{2}$$

yields

$$1 = \frac{1}{\sin^2 \frac{\pi}{2}} = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{\pi}{4}} + \frac{1}{\sin^2 \frac{3\pi}{4}}\right]$$

$$= \frac{1}{16}\left[\frac{1}{\sin^2 \frac{\pi}{8}} + \frac{1}{\sin^2 \frac{3\pi}{8}} + \frac{1}{\sin^2 \frac{5\pi}{8}} + \frac{1}{\sin^2 \frac{7\pi}{8}}\right] = \cdots$$

$$= \frac{1}{4^n}\sum_{k=0}^{2^n-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}} \tag{3}$$

$$= \frac{2}{4^n}\sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}}. \tag{4}$$

Taking the termwise limit $n \to \infty$ and using $\lim_{N\to\infty} N \sin(x/N) = x$ for $N = 2^n$ and $x = (2k+1)\pi/2$ yields the series

$$1 = \frac{8}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}, \tag{5}$$

from which (1) follows easily.

Now taking the limit termwise requires some care, as the example $1 = 1/2 + 1/2 = 1/4 + 1/4 + 1/4 + 1/4 = \cdots \to 0 + 0 + 0 \cdots = 0$ shows. In the above case (4) $\to$ (5) it is justified because the kth term in the sum (4) is bounded by $2/(2k+1)^2$ (independently of n) since $\sin x > 2x/\pi$ holds for $0 < x < \pi/2$. ■

The argument in the last step (i.e., interchanging limit and summation) is known as Tannery's Theorem (see [**16**, p. 292], [**5**], or [**4**]); we present it in an appendix at the end of this Note. It is instructive here to check that (and why) the termwise limit (3) $\to$ (5) fails.

Use of Tannery's Theorem can be avoided by the following elementary argument: Sum the inequalities $\sin^{-2} x > x^{-2} > \cot^2 x = \sin^{-2} x - 1$ (which follow from $\sin x < x < \tan x$ for $0 < x < \pi/2$) for $x = (2k+1)\pi/(2N)$ with $k = 0, \ldots, N/2 - 1$. Then (4) implies

$$1 > \frac{8}{\pi^2}\sum_{k=0}^{N/2-1} \frac{1}{(2k+1)^2} > 1 - \frac{1}{N},$$

for $N = 2^n$, and hence (5).

2. RELATED PROOFS. The proof in Section 1 was inspired by two related proofs (# 9 and # 10) among the 14 proofs of Euler's identity (1) collected by Chapman [**6**], and the identity

$$\sum_{k=0}^{N-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2N}} = N^2, \tag{6}$$

which I encountered 25 years ago as a mathematics olympiad problem [**2**]. A proof for (6) for general N is in Section 3. These two related proofs use instead the identities

$$\sum_{k=1}^{N} \cot^2 \frac{k\pi}{2N+1} = \frac{N(2N-1)}{3} \tag{7}$$

and

$$\sum_{k=1}^{N} \frac{1}{\sin^2 \frac{k\pi}{2N+1}} = \frac{2N(N+1)}{3}. \tag{8}$$

These identities (6)–(8) are usually proved by comparing the coefficients in a suitable polynomial of degree N whose zeroes are the terms of the sums. This way to prove (1) via (7) or (8) is described in detail in [**5**, ch. IX] (which also has (6)) and [**17**, ch. X], and was rediscovered in [**8**], [**12**], and [**13**].

The only new (?) feature in the present proof is the restriction to $N = 2^n$ where (6) allows a simpler argument.

For other (more or less) elementary proofs of (1) see [**1**], [**3**], [**6**], [**7**], [**9**], [**11**], [**12**], and [**15**], and references therein. There is an interesting historical account in [**15**].

3. THE PARTIAL FRACTION EXPANSION OF $\sin^{-2} x$. The identity (6) is a special case ($x = \pi/2$) of

$$\frac{1}{\sin^2 x} = \frac{1}{N^2} \sum_{k=0}^{N-1} \frac{1}{\sin^2 \frac{x+k\pi}{N}}. \tag{9}$$

This identity follows for $N = 2^n$ in the same way as in Section 1, starting from $\sin^{-2} x$. Writing it as

$$\frac{1}{\sin^2 x} = \frac{1}{N^2} \sum_{k=-N/2}^{N/2-1} \frac{1}{\sin^2 \frac{x+k\pi}{N}}$$

yields the partial fraction expansion of $\sin^{-2} x$ in the limit $N \to \infty$:

$$\frac{1}{\sin^2 x} = \sum_{k \in \mathbb{Z}} \frac{1}{(x+k\pi)^2}, \tag{10}$$

from which (9) can be verified for arbitrary N in turn. As pointed out by the referee, identity (8) can be derived from (9) by taking the limit $x \to 0$, and replacing N by $2N+1$.

This is a funny variation of Cauchy's original induction proof for the inequality of the arithmetic and geometric mean: To prove the identity (9) for arbitrary natural numbers N, we first prove it by an induction $n \to 2n$ for all powers of 2: $N = 2^n$. Then we take the limit $n \to \infty$ to obtain the infinite series (10), from which the formula follows for every finite N.

4. THE GREGORY–LEIBNIZ SERIES. The fact that

$$1 - \frac{1}{3} + \frac{1}{5} - + \cdots = \frac{\pi}{4} \tag{11}$$

can be proved in a similar fashion. We use the identity

$$\cot x = \frac{1}{2}\left[\cot\frac{x}{2} - \tan\frac{x}{2}\right] = \frac{1}{2}\left[\cot\frac{x}{2} - \cot\left(\frac{\pi - x}{2}\right)\right]$$

instead of (1). Then

$$\begin{aligned}
1 = \cot\frac{\pi}{4} &= \frac{1}{2}\left[\cot\frac{\pi}{8} - \cot\frac{3\pi}{8}\right] \\
&= \frac{1}{4}\left[\cot\frac{\pi}{16} - \cot\frac{7\pi}{16} - \cot\frac{3\pi}{16} + \cot\frac{5\pi}{16}\right] = \cdots \\
&= \frac{1}{N}\sum_{k=0}^{N-1}(-1)^k \cot\frac{(2k+1)\pi}{4N} \qquad (\text{for } N = 2^n).
\end{aligned}$$

Taking the limit $N \to \infty$ and using $(1/N)\cot(x/N) \to 1/x$ yields

$$1 = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}.$$

This series is not absolutely convergent. Still, Tannery's Theorem applies after combining two consecutive terms, e.g., using the formula $\cot\alpha - \cot\beta = \sin(\beta - \alpha)/\sin\alpha\sin\beta$.

More generally, the partial fraction expansion of $\cot x$ can be derived in a similar way; see [**10**, § 24] or [**14**].

Appendix: Tannery's Theorem. *If $s(n) = \sum_{k\geq 0} f_k(n)$ is a finite sum (or a convergent series) for each n, $\lim_{n\to\infty} f_k(n) = f_k$, $|f_k(n)| \leq M_k$, and $\sum_{k=0}^{\infty} M_k < \infty$ then*

$$\lim_{n\to\infty} s(n) = \sum_{k=0}^{\infty} f_k.$$

Proof. For any given $\varepsilon > 0$ there is an $N(\varepsilon)$ such that $\sum_{k>N(\varepsilon)} M_k < \varepsilon/3$. For each k there is an $N_k(\varepsilon)$ such that $|f_k(n) - f_k| < \varepsilon/(3N(\varepsilon))$ for all $n \geq N_k(\varepsilon)$. Let $\bar{N}(\varepsilon) = \max\{N_1(\varepsilon), \ldots, N_{N(\varepsilon)}(\varepsilon)\}$. Then

$$|s(n) - \sum_k f_k| \leq \sum_{k=0}^{N(\varepsilon)} |f_k(n) - f_k| + 2\sum_{k>N(\varepsilon)} M_k < N(\varepsilon)\frac{\varepsilon}{3N(\varepsilon)} + 2\frac{\varepsilon}{3} = \varepsilon$$

for all $n \geq \bar{N}(\varepsilon)$. ∎

A standard application of Tannery's Theorem is to show that the two usual definitions of e^x are the same:

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k} = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

Tannery's Theorem is related to the M-test of Weierstrass: *Let $f_k : D \to \mathbb{R}$ be a sequence of functions, $|f_k(x)| \leq M_k$, $\sum_k M_k < \infty$. Then $s(x) = \sum_k f_k(x)$ converges uniformly, and if each f_k is continuous, then s is continuous.*

With the domain $D = \{1, 2, \ldots, \infty\}$ the continuity at ∞ of f_k and s yields Tannery's Theorem.

Tannery's Theorem is also a special case of Lebesgue's dominated convergence theorem on the sequence space ℓ^1.

REFERENCES

1. M. Aigner and G. M. Ziegler, *Proofs from THE BOOK*, Springer-Verlag, Berlin, 1998.
2. Alpha **8** (3) (1974) 60.
3. T. M. Apostol, A proof that Euler missed: Evaluating $\zeta(2)$ the easy way, *Math. Intelligencer* **5** (1983) 59–60.
4. R. P. Boas, Tannery's theorem, *Math. Mag.* **38** (1965) 66.
5. T. J. l'A. Bromwich, *An Introduction to the Theory of Infinite Series*, 2nd ed., Macmillan, London, 1949.
6. R. Chapman, Evaluating $\zeta(2)$, Preprint, 1999, http://www.maths.ex.ac.uk/~rjc/etc/zeta2.dvi.
7. D. P. Giesy, Still another elementary proof that $\sum 1/k^2 = \pi^2/6$, *Math. Mag.* **45** (1972) 148–149.
8. F. Holme, En enkel beregning av $\sum_{k=1}^{\infty} \frac{1}{k^2}$, *Normat* **18** (1970) 91–92.
9. D. Kalman, Six ways to sum a series, *College Math. J.* **24** (1993) 402–421.
10. K. Knopp, *Theorie und Anwendung der unendlichen Reihen*, Springer-Verlag, Berlin, 1931.
11. K. Knopp and I. Schur, Über die Herleitung der Gleichung $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, *Archiv Math. Physik* **17** (1918) 174–176.
12. R. A. Kortram, Simple proofs for $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ and $\sin x = x \prod_{k=1}^{\infty}(1 - \frac{x^2}{k^2\pi^2})$, *Math. Mag.* **69** (1996) 122–125.
13. I. Papadimitriou, A simple proof of the formula $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$, *Amer. Math. Monthly* **80** (1973) 424–425.
14. H. Schröter, Ableitung der Partialbruch- und Produkt-Entwicklungen für die trigonometrischen Funktionen, *Z. Math. Physik* **13** (1868) 254–259.
15. P. Stäckel, Eine vergessene Abhandlung Leonhard Eulers über die Summe der reziproken Quadrate der natürlichen Zahlen, *Bibliotheka mathematica* **III (8)** (1907) 37–60.
16. J. Tannery, *Introduction a la Théorie des Fonctions d'une Variable*, 2 ed., Tome 1, Libraire Scientifique A. Hermann, Paris, 1904.
17. A. M. Yaglom and I. M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions, Vol. II*, Dover, New York, 1967.

Universität Wien, A–1090 Vienna, Austria
Josef.Hofbauer@univie.ac.at